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Null Strings in Kerr Spacetime ¹

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Abstract

The null string's equations of motion and constraints in the Kerr spacetime are given. We assume a generic ansatz for the null strings in the Kerr spacetime and we present the resulting solutions in quadratures. Some specific string configurations, that follow from the generic one, are considered separately. In each case we also extract the corresponding solutions in the Schwarzschild spacetime.

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1 Introduction

The analysis of the classical string equations of motion and constraints in generic curved backgrounds [1] has become an active area of research [2, 3, 5, 4, 6, 7, 8]. The study of classical strings in curved spacetime backgrounds provides us with better understanding of the physics of gravitation in the context of string theory.

It is well known that the classical evolution of strings is described by a complicated system of second-order non-linear coupled partial differential equations [9]. In order to properly analyse the motion of classical strings in gravitation backgrounds we need the exact solutions of these equations which can be obtained by choosing a specific ansatz. Different forms of symmetric ansatz have been proposed. In these, the string evolution equations become ordinary differential equations.

In the case of null strings the situation is further simplified since the null strings are similar to the massless point particles. Schild [10] was the first one who introduced the notion of a null or tensionless string, (see also [11]). The first reference on null strings in a curved background can be found in the concluding portion of Schild's paper. The null strings can be considered as a zero approximation with the string tension as the perturbation parameter [12, 13]. Calculations involving null strings in different metrics have been performed recently [14, 15, 16].

The task of this paper is to discuss the null string evolution in the Kerr spacetime. In Section 2 we present the general equations of motion for null strings in the Kerr spacetime. Next in Section 3 we give the general solutions in quadratures in the case of the generic ansatz for the null strings. In Section 4 we consider the particular case of a circular null string and the corresponding solutions in the Schwarzschild spacetime. In Section 5 we do the same thing for another interesting particular case arising from the generic ansatz. Finally in Section 6, we summarize our results and give some concluding remarks.

2 Null Strings in the Kerr Spacetime

Let us consider a generic background metric given by

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + 2g_{03}dtd\phi + g_{33}d\phi^2 \quad (1)$$

The tensor g_{ij} for the Kerr background spacetime is given by

$$\begin{aligned}
g_{00} &= \frac{2mr}{r^2 + a^2 \cos^2 \theta} - 1, \\
g_{03} &= -\frac{2mar \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}, \\
g_{11} &= \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2}, \\
g_{22} &= r^2 + a^2 \cos^2 \theta, \\
g_{33} &= \left(r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \right) \sin^2 \theta,
\end{aligned}$$

where m, a are the mass and the angular momentum per unit mass respectively. The Kerr black hole has two coordinate singularities corresponding to the outer and inner horizon $r_{\pm} = m \pm \sqrt{m^2 - a^2}$, for $m^2 > a^2$. We follow the notations given in [17, 18]

Let us consider the null string equations of motion

$$\ddot{X}^\mu + \Gamma_{\nu\rho}^\mu \dot{X}^\nu \dot{X}^\rho = 0. \quad (2)$$

The constraints read as

$$\begin{aligned}
g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu &= 0, \\
g_{\mu\nu} \dot{X}^\mu X'^\nu &= 0.
\end{aligned} \quad (3)$$

where the overdots and primes denote differentiations with respect to τ and σ , respectively. We see that for the null strings we have the null geodesic equations [17, 18] supplemented by the second constraint in the (3). From the second constraint we see that each point of the string propagates in the direction perpendicular to the string. We notice that the constraint equations are not invariant under arbitrary reparametrizations of σ and τ . It is easy to see that equations (2) and constraints (3) are invariant only under transformations of the form $\tau_1 = f(\tau, \sigma)$ and $\sigma_1 = g(\sigma)$ where $f(\tau, \sigma)$ and $g(\sigma)$ are arbitrary differentiable functions.

We use the following notation for the coordinates of a null string

$$X^0 = t(\tau, \sigma), \quad X^1 = r(\tau, \sigma), \quad X^2 = \theta(\tau, \sigma), \quad X^3 = \phi(\tau, \sigma). \quad (4)$$

The equations of motion (2) for a null string moving in the Kerr spacetime become now

$$\begin{aligned}
\ddot{t} + 2\Gamma_{01}^0 \dot{t}\dot{r} + 2\Gamma_{02}^0 \dot{t}\dot{\theta} + 2\Gamma_{13}^0 \dot{r}\dot{\phi} + 2\Gamma_{23}^0 \dot{\theta}\dot{\phi} &= 0 \\
\ddot{\phi} + 2\Gamma_{01}^3 \dot{t}\dot{r} + 2\Gamma_{02}^3 \dot{t}\dot{\theta} + 2\Gamma_{13}^3 \dot{r}\dot{\phi} + 2\Gamma_{23}^3 \dot{\theta}\dot{\phi} &= 0 \\
\ddot{r} + \Gamma_{00}^1 \dot{t}^2 + 2\Gamma_{03}^1 \dot{t}\dot{\phi} + \Gamma_{11}^1 \dot{r}^2 + 2\Gamma_{12}^1 \dot{r}\dot{\theta} + \Gamma_{22}^1 \dot{\theta}^2 + \Gamma_{33}^1 \dot{\phi}^2 &= 0 \\
\ddot{\theta} + \Gamma_{00}^2 \dot{t}^2 + 2\Gamma_{03}^2 \dot{t}\dot{\phi} + \Gamma_{11}^2 \dot{r}^2 + 2\Gamma_{12}^2 \dot{r}\dot{\theta} + \Gamma_{22}^2 \dot{\theta}^2 + \Gamma_{33}^2 \dot{\phi}^2 &= 0
\end{aligned} \tag{5}$$

and the constraints (3) can be expressed as follows, using the notation (4):

$$\begin{aligned}
g_{00}\dot{t}^2 + 2g_{03}\dot{t}\dot{\phi} + g_{11}\dot{r}^2 + g_{22}\dot{\theta}^2 + g_{33}\dot{\phi}^2 &= 0 \\
g_{00}\dot{t}t' + g_{03}(\dot{t}\phi' + \dot{\phi}t') + g_{11}\dot{r}r' + g_{22}\dot{\theta}\theta' + g_{33}\dot{\phi}\phi' &= 0
\end{aligned} \tag{6}$$

where Γ_{jk}^i and g_{ij} are functions from the r , θ . Making the following substitution:

$$u = \dot{t} \quad , \quad v = \dot{\phi} \tag{7}$$

in the (5) we obtain:

$$\begin{aligned}
\dot{u} + 2(\Gamma_{01}^0 u + \Gamma_{13}^0 v)\dot{r} + 2(\Gamma_{02}^0 u + \Gamma_{23}^0 v)\dot{\theta} &= 0 \\
\dot{v} + 2(\Gamma_{01}^3 u + \Gamma_{13}^3 v)\dot{r} + 2(\Gamma_{02}^3 u + \Gamma_{23}^3 v)\dot{\theta} &= 0 \\
2g_{11}\ddot{r} + g_{11,1}\dot{r}^2 + 2g_{11,2}\dot{r}\dot{\theta} - g_{22,1}\dot{\theta}^2 - (g_{00,1}u^2 + 2g_{03,1}uv + g_{33,1}v^2) &= 0 \\
2g_{22}\ddot{\theta} + g_{22,2}\dot{\theta}^2 + 2g_{22,1}\dot{r}\dot{\theta} - g_{11,2}\dot{r}^2 - (g_{00,2}u^2 + 2g_{03,2}uv + g_{33,2}v^2) &= 0
\end{aligned} \tag{8}$$

and the constraints (6) now read

$$\begin{aligned}
g_{11}\dot{r}^2 + g_{22}\dot{\theta}^2 + g_{00}u^2 + 2g_{03}uv + g_{33}v^2 &= 0 \\
g_{00}ut' + g_{03}(u\phi' + vt') + g_{11}\dot{r}r' + g_{22}\dot{\theta}\theta' + g_{33}v\phi' &= 0
\end{aligned} \tag{9}$$

The first two equations in (5) cannot be integrated in general, although this can be done in the case of Schwarzschild spacetime. We will try to investigate the existence of solution when a generic ansatz for null strings in the Kerr spacetime is employed.

3 Ansatz

In this section we consider the following generic ansatz for a null string in the Kerr spacetime

$$t = \gamma\sigma + t_1(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = \beta\sigma + \phi_1(\tau). \quad (10)$$

which helps us simplify considerably the systems of equations (8) and (9). The variables τ and σ are respectively the time-like and space-like coordinates on the worksheet and γ , β are constants. We obtain from (8) the system equations

$$u = C_1(\beta g_{33} + \gamma g_{03}) \exp\{-G(r, \theta)\} \quad (11)$$

$$v = -C_1(\gamma g_{00} + \beta g_{03}) \exp\{-G(r, \theta)\} \quad (12)$$

$$2g_{11}\ddot{r} + g_{11,1}\dot{r}^2 + 2g_{11,2}\dot{r}\dot{\theta} - g_{22,1}\dot{\theta}^2 - (g_{00,1}u^2 + 2g_{03,1}uv + g_{33,1}v^2) = 0 \quad (13)$$

$$2g_{22}\ddot{\theta} + g_{22,2}\dot{\theta}^2 + 2g_{22,1}\dot{r}\dot{\theta} - g_{11,2}\dot{r}^2 - (g_{00,2}u^2 + 2g_{03,2}uv + g_{33,2}v^2) = 0 \quad (14)$$

$$g_{11}\dot{r}^2 + g_{22}\dot{\theta}^2 + g_{00}u^2 + 2g_{03}uv + g_{33}v^2 = 0 \quad (15)$$

$$(\gamma g_{00} + \beta g_{03})u + (\gamma g_{03} + \beta g_{33})v = 0 \quad (16)$$

where C_1 is constant since u and v are function only τ . The function $G(r, \theta)$ in the exponent have the form

$$G(r, \theta) = \int g^{00}dg_{00} + 2 \int g^{03}dg_{03} + \int g^{33}dg_{33} \quad (17)$$

Multiplying (13) by \dot{r} , (14) by $\dot{\theta}$, summing the results together and subtracting the derivative with respect to τ of (15) we get

$$\frac{dg}{d\tau} - g \frac{dG}{d\tau} = 0 \quad (18)$$

where $g = g_{00}g_{33} - g_{03}^2$. After integrating the equation (18) we have

$$G = \ln(g) \quad (19)$$

In (17) and (19) we omitted the constants of integration which are included in the definition of C_1 .

Now we integrate the equation (14). Multiplying (14) by $g_{22}\dot{\theta}$ we have

$$\frac{d}{d\tau}(g_{22}^2\dot{\theta}^2) + K(r, \theta)\frac{d\theta}{d\tau} = 0 \quad (20)$$

where

$$K(r, \theta) = \frac{C_1^2}{g^2} \left[g \frac{\partial g' g_{22}}{\partial \theta} - g' g_{22} \frac{\partial g}{\partial \theta} \right] \quad (21)$$

and $g' = \gamma^2 g_{00} + 2\gamma\beta g_{03} + \beta^2 g_{33}$. It is now obvious that this equation can be integrated. So finally after having substituted (19) in (11) and (12), we get the following system equations for the ansatz (10)

$$\begin{aligned} \dot{t}_1 &= \frac{C_1(\beta g_{33} + \gamma g_{03})}{g} \\ \dot{\phi}_1 &= -\frac{C_1(\gamma g_{00} + \beta g_{03})}{g} \\ \dot{r}^2 &= -\frac{C_1^2 g'}{g g_{11}} - g^{22} g^{11} [C_2 - a^2 C_1^2 \beta^2 \sin^2 \theta - C_1^2 \gamma^2 \sin^{-2} \theta] \\ g_{22}^2 \dot{\theta}^2 &= C_2 - a^2 C_1^2 \beta^2 \sin^2 \theta - C_1^2 \gamma^2 \sin^{-2} \theta \end{aligned} \quad (22)$$

Notice that if we define the invariant string size as

$$S(\tau) = \int_0^{2\pi} \sqrt{g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} d\sigma \quad (23)$$

we have from equations (22) and for the Kerr metric

$$S(\tau) = 2\pi \sqrt{\gamma^2 g_{00} + 2\gamma\beta g_{03} + \beta^2 g_{33}} \quad (24)$$

In the case $\theta = \text{const.}$ we have $C_2 = a^2 C_1^2 \beta^2 \sin^2 \theta + C_1^2 \gamma^2 \sin^{-2} \theta$ and from the system of equations (22) follows that

$$\begin{aligned} \dot{t}_1 &= C_1(\beta g^{00} - \gamma g^{03}) \\ \dot{\phi}_1 &= C_1(\beta g^{03} - \gamma g^{33}) \\ \dot{r}^2 &= -C_1^2 g^{11} (\beta^2 g^{00} - 2\gamma\beta g^{03} + \gamma^2 g^{33}) \end{aligned} \quad (25)$$

Now we can integrate the set of equations (25) completely and we obtain

$$\begin{aligned}\pm(t_1 - t_1^0) &= \int_r^{r_0} (\beta g^{00} - \gamma g^{03}) [g^{11} (2\gamma \beta g^{03} - \beta^2 g^{00} - \gamma^2 g^{33})]^{-\frac{1}{2}} dr \\ \pm(\phi_1 - \phi_1^0) &= \int_r^{r_0} (\beta g^{03} - \gamma g^{33}) [g^{11} (2\gamma \beta g^{03} - \beta^2 g^{00} - \gamma^2 g^{33})]^{-\frac{1}{2}} dr\end{aligned}\tag{26}$$

We can also derive from (25) the interesting case of a null string moving in the Schwarzschild spacetime. The metric can be taken from the Kerr's one by putting $a = 0$. Then the system of equations for the generic ansatz (10) are the following

$$\begin{aligned}\dot{t}_1 &= C_1 \beta g^{00} \\ \dot{\phi}_1 &= -C_1 \gamma g^{33} \\ \dot{r}^2 &= C_1^2 \beta^2 - C_2 g^{22} g^{11} \\ g_{22}^2 \dot{\theta}^2 &= C_2 - C_1^2 \gamma^2 \sin^{-2} \theta\end{aligned}\tag{27}$$

4 Circular Null Strings

It is easy to see that by taking $\gamma = 0$ and $\beta = 1$ in the generic ansatz (10) we obtain the particular configuration of a circular null string moving in Kerr spacetime

$$t = t(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = \sigma + \phi_1(\tau).\tag{28}$$

The system equation for this particular string configuration can be extracted from the more general ones (22), after substituting the values of the parameters γ and β . It is checked, however, that the same equations can be obtained directly from the equations of motion by taking into the account the specific ansatz (28).

$$\begin{aligned}\dot{t} &= C_1 g^{00} \\ \dot{\phi}_1 &= C_1 g^{03} \\ \dot{r}^2 &= -C_1^2 g^{00} g^{11} - g^{22} g^{11} (C_2 - a^2 C_1^2 \sin^2 \theta) \\ g_{22}^2 \dot{\theta}^2 &= C_2 - a^2 C_1^2 \sin^2 \theta\end{aligned}\tag{29}$$

Since $\dot{\phi}_1$ does not vanish, ϕ is a function of both σ and τ . When $\theta = \text{const.}$ we have $C_2 = a^2 C_1^2 \sin^2 \theta$ and

$$\begin{aligned}\pm(t - t_0) &= \int_r^{r_0} \sqrt{-g^{00} g_{11}} dr \\ \pm(\phi_1 - \phi_1^0) &= \int_r^{r_0} g^{03} \sqrt{-\frac{g_{11}}{g^{00}}} dr\end{aligned}\tag{30}$$

We return now to the case of Schwarzschild spacetime, and consider again the evolution of a circular null string. The system equations (29) now takes the form:

$$\begin{aligned}\dot{t} &= C_1 g^{00} \\ \dot{\phi}_1 &= 0 \\ \dot{r}^2 &= C_1^2 - C_2 g^{22} g^{11} \\ g_{22}^2 \dot{\theta}^2 &= C_2\end{aligned}\tag{31}$$

In this case $\dot{\phi}_1$ vanishes and thus ϕ is a function only of σ . Consequently, in the case of Schwarzschild background metric, a possible ansatz for circular null string is given by

$$t = t(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = \sigma\tag{32}$$

5 Another possible dynamic string configuration

Let us now interchange the values between the parameters γ and β , i.e. let us put $\beta = 0$ and $\gamma = 1$. We are then led to the following ansatz

$$t = \sigma + t_1(\tau), \quad r = r(\tau), \quad \theta = \theta(\tau), \quad \phi = \phi(\tau).\tag{33}$$

Following Section 4, we can substitute in the results of Section 3 these particular values of the parameters and obtain the following system of ordinary differential equations for this particular string configuration

$$\begin{aligned}
\dot{t}_1 &= -C_1 \gamma g^{03} \\
\dot{\phi} &= -C_1 \gamma g^{33} \\
\dot{r}^2 &= -C_1^2 g^{11} g^{33} - g^{22} g^{11} [C_2 - C_1^2 \gamma^2 \sin^{-2} \theta] \\
g_{22}^2 \dot{\theta}^2 &= C_2 - C_1^2 \gamma^2 \sin^{-2} \theta
\end{aligned} \tag{34}$$

When $\theta = \text{const.}$ for the Kerr spacetime we have $C_2 = C_1^2 \gamma^2 \sin^{-2} \theta$ and

$$\begin{aligned}
\pm (t_1 - t_1^0) &= \int_r^{2m} g^{03} \sqrt{-\frac{g_{11}}{g^{33}}} dr \\
\pm (\phi - \phi_0) &= \int_r^{2m} \sqrt{-g^{11} g^{33}} dr
\end{aligned} \tag{35}$$

The system equations (35) is meaningful only when $g^{11} g^{33} < 0$. For the case $\theta = \frac{\pi}{2}$ we have that $g^{11} g^{33} = \frac{r-2m}{r^3}$ and from this follows that at the equatorial plane we have solution only in the region $r < 2m$.

If the case of Schwarzschild spacetime the equations (34) take the form

$$\begin{aligned}
\dot{t}_1 &= 0 \\
\dot{\phi} &= -C_1 \gamma g^{33} \\
\dot{r}^2 &= -C_1^2 g^{11} g^{33} - g^{11} g^{22} [C_2 - C_1^2 \gamma^2 \sin^{-2} \theta] \\
g_{22}^2 \dot{\theta}^2 &= C_2 - C_1^2 \gamma^2 \sin^{-2} \theta
\end{aligned} \tag{36}$$

It is easy to see that when $\theta = \text{const.}$ these equations do not have any solution outside the horizon of the Schwarzschild black hole. This is the reason why we cannot have an ansatz for the string in the form

$$t = \gamma \sigma, \quad r = r(\tau), \quad \theta = \text{const.}, \quad \phi = \phi(\tau) \tag{37}$$

outside the horizon in the case of the Schwazchild spacetime.

6 Conclusion

To conclude, let us now summarize the results we obtained. We have written down the null string equations of motion and constraints for the case of the Kerr spacetime. We have been able to construct the general solutions in

quadratures for a generic ansatz for the null string and we also gave the system of equations for the case of Schwarzschild spacetime.

We considered also the particular case of the evolution of a circular null string in Kerr spacetime. Here too the solution is given in quadratures. The specific case of the Schwarzschild spacetime is also discussed and it agrees with other authors [14, 15]. Another null string configuration determined by the ansatz (33) is also examined. For the special of the equatorial plane it describes the motion of the string for $r < 2m$.

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